

Module 7 Lebesgue measure on \mathbb{R}^n

7.1 Lebesgue measure in \mathbb{R}^n [Fol 2.6]

今日: Lebesgue measure in \mathbb{R}^n 的

- regularity
- behavior under affine transformation
- behavior under diffeomorphism

7.1.1 Lebesgue measure in \mathbb{R}^n

这是 product measure 最常见的应用和例子.

Def 7.1

$(\mathbb{R}^n, \mathcal{L}^n, m)$ Lebesgue measure is **completion of** $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m|_{\text{borel}})$.



where $\mathcal{B}_{\mathbb{R}^n} = \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}$ $\mathcal{L}^n = \{\text{Leb meas sets}\} \supset \mathcal{B}_{\mathbb{R}^n}$ Write:

$$\int f dm^n$$

Theorem 7.1 (Fubini-Tonelli for m^n)

Suppose $f \in L^+(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n)$

$$\int f dm^n = \int \cdots \int f(x_1, \dots, x_n) dx_1 \cdots dx_n \quad (7.1)$$

$$= \int \cdots \int f(x_1, \dots, x_n) dx_n \cdots dx_1 \quad (7.2)$$



Example 7.1 Show:

$$\int_0^\infty e^{-sx} \frac{\sin^2(x)}{x} dx = \frac{1}{4} \log(1 + 4s^{-2})$$

for $s > 0$, by integrating $e^{-sx} \sin 2xy = f(x, y)$ over the rectangle $x \in (0, \infty), y \in (0, 1)$.

Sketch: $f \in L^1$ (since it is ctn on \mathbb{R}) 以及

$$|f| \leq e^{-sx}, \quad \int_{\mathbb{R}} e^{-sx} < \infty$$

可计算得

$$\int_0^1 \sin 2xy dy = \frac{1}{2x} \sin^2 x$$

而后 compute

$$\int_0^1 e^{-sx} \sin 2xy dy$$

by integration by part for twice.

7.1.2 regularities of Lebesgue measure in \mathbb{R}^n **Theorem 7.2 (regularities of \mathcal{L}^n)**

If $E \subset \mathcal{L}^n$, 则有:

- **outer regularity:**

$$m(E) = \inf\{m(U) \mid U \text{ open } \supset E\}$$

- **inner regularity:**

$$m(E) = \sup\{m(K) \mid K \text{ compact } \subset E\}$$

- if $m(E) < \infty$, 则对于任意 $\epsilon > 0$, 都存在 disjoint rectangles R_1, \dots, R_N with sides that are open intervals (literally rectangles) s.t.

$$m(E \Delta \bigcup_j R_j) < \epsilon$$

**Proof for (a,b) i.e. regularities:**

Fix $\epsilon > 0$. By construction, 存在 finite disjoint union of rectangle T_j for each j , 使得

$$E \subset \bigcup_{j=1}^{\infty} T_j \quad \text{and} \quad \sum_{j=1}^{\infty} m(T_j) \leq m(E) + \epsilon$$

By outer regularity of m^1 , 存在 $U_j \supset T_j$ open rect s.t. $m(U_j) \leq m(T_j) + \epsilon/2^j$ Then:

$$E \subset U := \bigcup_{j=1}^{\infty} U_j \quad \text{and} \quad m(U) \leq \sum_{j=1}^{\infty} m(U_j)$$

Construct K as in dim 1 (DIY) $\leq m(E) + 2\epsilon$.

(完整 Pf 可见 395 笔记, 此略)

Proof for (c):

Notation as above.

$$m(E) < \infty \implies m(U) < \infty \implies m(U_j) < \infty \quad \forall j$$

Sides of U_j are disjoint union of ctbly many open finite intervals.

因而存在 open rectangle $V_j \subset U_j$ for each j that are finite disjoint union of finite open intervals s.t.

$$m(U_j \setminus V_j) < \epsilon/2^j$$

Now pick R_1, \dots, R_N from honest rectangles (即 sides 都是 intervals 的 rectangle) insides V_j (DIY). (完整 Pf 可见 395 笔记, 此略)

Corollary 7.1

For $f \in L^1(m)$, if $f \in L^1(m)$ and $\epsilon > 0$ then

- 对于任意 $\epsilon > 0$, 都存在 $\phi = \sum_{j=1}^N c_j \chi_{R_j}$ s.t.

$$\int |\phi - f| dm < \epsilon$$

其中 each $c_j \in \mathbb{C}$, R_j 是 rectangles with sides as finite open intervals.

- 存在 $\phi \in C_c^0(\mathbb{R}^n)$ s.t.

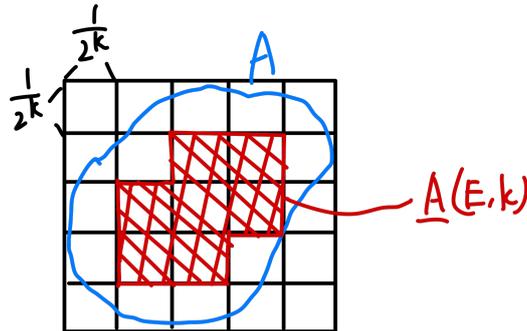
$$\int |f - \phi| dm < \epsilon$$



Proof Similar to 1 dim case, 可以证明 {all step functions}, $C_c^0(\mathbb{R}^n)$ 是 dense subspace of $L^1(m)$.

7.1.3 approximating an open set $E \subset \mathbb{R}^n$ by countable disjoint interior cubes

对于 $k \in \mathbb{Z}$, 令 \mathcal{Q}_k be the collection of cubes whose side length is $\frac{1}{2^k}$ 且 vertices 在 lattice $(2^{-k}\mathbb{Z})^n$ 中, 即精细度为 $\frac{1}{2^k}$ 的网格中的所有 cubes.



对于 $E \subset \mathbb{R}^n$, 我们定义:

$$\underline{A}(E, k) := \bigcup \{Q \in \mathcal{Q}_k : Q \subset E\}, \quad \overline{A}(E, k) := \bigcup \{Q \in \mathcal{Q}_k : Q \cap E \neq \emptyset\}$$

即, 一个是被包含在 E 中的所有格子, 一个是最小的覆盖 E 的所有格子. 并定义:

$$\underline{A}(E) := \bigcup_{k=1}^{\infty} \underline{A}(E, k), \quad \overline{A}(E) := \bigcup_{k=1}^{\infty} \overline{A}(E, k)$$

以及

$$\overline{\kappa}(E) := \lim_{k \rightarrow \infty} m(\underline{A}(E, k)), \quad \underline{\kappa}(E) := \lim_{k \rightarrow \infty} m(\overline{A}(E, k))$$

By CFB, CFA 容易得到:

$$\overline{\kappa}(E) = m(\overline{A}(E)), \quad \underline{\kappa}(E) = m(\underline{A}(E))$$

Note: 这里的 $\underline{A}(E, k)$, $\overline{A}(E, k)$, $\underline{A}(E)$, $\overline{A}(E)$ 都是 union of cubes with disjoint interiors.

Lemma 7.1 (approximate an open set by disjoint interior cubes)

Let $E \subset \mathbb{R}^n$ be open.

Claim: $E = \underline{A}(E)$



Proof Folland 2.43.

Corollary 7.2

$E \subset \mathbb{R}^n$ 是 Lebesgue measurable 的 $\iff \overline{\kappa}(E) = \underline{\kappa}(E)$



7.1.4 behavior under affine transformation

Affine transformation 即 linear transformation + translation.

7.1.5 Lebesgue measure and integral is invariant under translation

对于 $a \in \mathbb{R}^n$, 一个 translation $t : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto x + a$ 是 ctn 的并且

$$t_a^{-1} = t_{-a}$$

Theorem 7.3 (Lebesgue measure and integral is invariant under translation)

(a) 任取 $a \in \mathbb{R}^n$,

$$E \in \mathcal{L}^n \implies t_a(E) \in \mathcal{L}^n \quad \text{and} \quad m(t_a(E)) = m(E)$$

(b) if $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is Leb measurable, then so is $f \circ t_a$.

More, if $f \in L^+$ or $f \in L^1$, then $f \circ t_a \in L^1$ 并且

$$\int (f \circ t_a) dm = \int f dm$$



Remark 集合的 measure 以及 measurable function 的积分在 translation 下保持不变.

Proof (Folland 2.42)

(a) t_a ctn $\implies t_a(\mathcal{B}_{\mathbb{R}^n}) \subset \mathcal{B}_{\mathbb{R}^n}$, 因而 $t_a(\mathcal{B}_{\mathbb{R}^n}) = \mathcal{B}_{\mathbb{R}^n}$ E rectangle, so $E = E_1 \times \cdots \times E_n$, each in $\mathcal{B}_{\mathbb{R}}$
 $m(E) = \prod_1^n m(E_i), t_a(E) = \prod t_{a_i}(E_i)$ 因而

$$m(t_a(E)) = \prod m(t_{a_i}(E_i)) = \prod m(E_i) \subset m(E)$$

BY HK uniqueness, get

$$m(t_a(E)) = m(E) \quad \forall E \in \mathcal{B}_{\mathbb{R}^n}$$

if $N \subset \mathbb{R}^n$ subnull set, so is $t_a(N)$. 因而

$$m(t_a(E)) = m(E) \quad \forall E \in \mathcal{L}^n$$

(b) Pick $B \in \mathcal{B}_{\mathbb{C}} \implies f^{-1}(B) \in \mathcal{L}$. 因而 $f^{-1}(B) = E \cup N, E \in \mathcal{B}_{\mathbb{R}^n}, N$ null set 因而

$$(f \circ t_a)^{-1}(B) = t_a^{-1}(f^{-1}(B)) \tag{7.3}$$

$$= t_a^{-1}(E) \cup t_a^{-1}(N) \text{ (one Borel, one null)} \tag{7.4}$$

$$= t_{-a}(f^{-1}(B)) \tag{7.5}$$

当 $f = \chi_E$ 时, 积分 reduce to measure, 即 (a); 因而

$$\int (f \circ t_a) dm = \int f dm$$

also holds for simple f , by linearity.

从而 by def, 也 hold for $f \in L^+$ 和 $f \in L^1$.

7.1.6 Lebesgue measure and integration is scaled $|\det T|$ under linear map

Theorem 7.4 (Lebesgue measure and integration is scaled $|\det T|$ by linear map)

For $T \in GL(n, \mathbb{R})$ (即 linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ 且可逆) (a) 如果 $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is Lebesgue measurable, then so is $f \circ T$.

Moreover if $f \in L^+$ or $f \in L^1$, then $f \circ T \in L^+$, $f \circ T \in L^1$ respectively. And

$$\int f \, dm = |\det T| \int f \circ T \, dm$$

(b)

$$E \in \mathcal{L}^n \implies T(E) \in \mathcal{L}^n \quad \text{and} \quad m(T(E)) = |\det T| m(E)$$



Proof Note: 对于 $T, S \in GL(n, \mathbb{R})$, 如果

$$\int f = |\det T| \int f \circ T \quad \text{and} \quad \int f = |\det S| \int f \circ S$$

, 那么则有

$$\int f = |\det(T \circ S)| \int f \circ (T \circ S)(x)$$

which trivially follows from computation. (and $\det(S \circ T) = \det S \times \det T$ for any linear map S, T .)

recall that:

Lemma 7.2 (row reduction)

任意 invertible linear map 可以被拆分为 finite 个 elementary linear maps. (T_1 : scale 一行; T_2 : 交换两行; T_3 : 一行加上另一行的倍数).



于是, 我们只需要 prove the theorem for elementary linear maps 就可以了. 而 elementary linear maps 的 cases 则 easily follows from Fubini-Toneilli.

Let f be Borel measurable.

对于 T_2 : 交换两行 (其 \det 为 -1), 我们改变 the order of integration for two coordinates, 因而 integration 不变;

对于 T_1 : scale 一行 by const c (其 \det 为 c), 我们在一个 coordinate 上积分值翻 c 倍, 因而整体积分值翻 c 倍. 这里用到了 $\mathbb{R} \rightarrow \mathbb{R}$ 的 Lebesgue integral 的已证明结论:

$$\int f(t) \, dt = |c| \int f(ct) \, dt$$

对于 T_3 : 一行加上另一行的倍数 (其 \det 为 1), 我们 recall $\mathbb{R} \rightarrow \mathbb{R}$ 的 Lebesgue integral 的 translation invariance:

$$\int f(t+a) \, dt = \int f(t) \, dt$$

因而整体积分值不变.

从而我们证明了 (a) for Borel measurable f .

从而, (b) for Borel set E trivially follows from (a), by taking indicator function.

而对于 (b) 的 E Lebesgue measurable case, $E = B \cup N$ for some Borel set B 以及 subnull set N , 从而 $m(E) = m(B)$.

从而 (b) proved.

而 (a) 的 f Lebesgue measurable 的 case, by def reduces to $f = \chi_E$ where E is Lebesgue measurable set, 于是 follows from the (b).

7.1.7 Lebesgue measure is invariant under rotation (and reflection)

Corollary 7.3 (Lebesgue measure is invariant under rotation)

对于 rotation 和 reflection (即 orthogonal transformation), 即 $TT^* = I_n$ 的 linear map T , 有 $m(T(E)) = m(E)$.



Proof $TT^* = I_n \implies |\det(T)| = 1$.

Remark $A \in GL(n, \mathbb{R})$ 为一个 orthogonal transformation (可写作 $A \in O(n)$) 的定义是它 preserve norm. 我们知道, $A \in O(n)$ 当且仅当 $A^* = A^{-1}$.

有两种情况: rotation ($\det A = 1$) 和 reflection ($\det A = -1$).

7.2 Change of Variable Thm on \mathbb{R}^n [Fol 2.6, finished]

7.2.1 COV

Theorem 7.5 (general change of variable theorem)

Suppose $\Omega \subset \mathbb{R}^n$ open, $G : \Omega \rightarrow \mathbb{R}^n$ 为一个 C^1 diffeomorphism.

Claim:

- (a) 如果 $f : G(\Omega) \rightarrow \mathbb{C}$ 上是 Lebesgue measurable 的, 则 $f \circ G : \Omega \rightarrow \mathbb{C}$ 也是 Lebesgue measurable 的. 并且, 如果 $f \in L^+(G(\Omega), m)$ 或者 $f \in L^1(G(\Omega), m)$, 则有

$$\int_{G(\Omega)} f \, dm = \int_{\Omega} (f \circ G) |\det DG| \, dm$$

- (b) 如果 $E \subset \Omega$ 是 Lebesgue measurable set, 则 $G(E)$ 也是 Lebesgue measurable set, 并且

$$m(G(E)) = \int_E |\det DG| \, dm$$



Proof 首先, 类似于上一个 lecture 中的各个证明, 只需要 prove for Borel measurable functions 和 Borel sets 就可以了. 我们分为五步证明.

Step 1: 我们首先证明, 在 E 为一个 closed cube 的情况下 (我们转面用 Q 来表示它), 有

$$m(G(Q)) \leq \int_Q |\det DG(x)| \, dx$$

Proof of Step 1:

$$Q = \{x : \|x - a\|_{\text{sup}} \leq h\}$$

By MVT 容易得到, 对于任意的 $x \in Q$, 有:

$$\|G(x) - G(a)\|_{\text{sup}} \leq h \cdot (\sup_{y \in Q} \|DG(y)\|_{\text{sup}})$$

(by bounding each entry.)

从而, 我们发现 $G(Q)$ 是 contained in 一个边长是 $h \cdot \sup_{y \in Q} \|DG(y)\|_{\text{sup}}$ 的 cube 的.

从而有:

$$m(G(Q)) \leq (\sup_{y \in Q} \|DG(y)\|_{\text{sup}})^n m(Q)$$

在 invertible T 的作用下, $T^{-1} \circ G$ 仍然是一个 diffeomorphism, 从而

$$m(G(Q)) = |\det T| m(T^{-1}(G(Q))) \quad (7.6)$$

$$\leq |\det T| (\sup_{y \in Q} \|T^{-1} DG(y)\|)^n m(Q) \quad (7.7)$$

Let $\epsilon > 0$.

由于 DG 是 continuous 的, $DG(x)^{-1} DG(y)$ 也是 ctn 的 (从而 **uni.ctn.** in the compact cube), 我们对于任意 $\epsilon > 0$ 都可以找到一个 $\delta > 0$ 使得 对于任意的 $y, z \in Q$ s.t. $\|y - z\|_{\text{sup}} \leq \delta$, 都有

$$\|DG(x)^{-1} DG(y)\| \leq 1 + \epsilon$$

于是我们可以把 Q 切分成 interior disjoint 的 closed subcubes Q_1, \dots, Q_N , 标记其各个中心为 x_1, \dots, x_N , 其每个的 side length 都至多为 δ , 从而有 $G(Q) \subset \bigcup_{j=1}^N m(G(Q_j))$. 于是

$$m(G(Q)) \leq \sum_{j=1}^N m(G(Q_j)) \quad (7.8)$$

$$\leq \sum_{j=1}^N |\det DG(x_j)| (\sup_{y \in Q_j} \|DG(x_j)^{-1} DG(y)\|_{\text{sup}})^n m(Q_j) \quad (7.9)$$

$$\leq (1 + \epsilon) \sum_{j=1}^N |\det DG(x_j)| m(Q_j) \quad (7.10)$$

$$\rightarrow (1 + \epsilon) |\det DG(x)| m(Q) \quad \text{as } \delta \rightarrow 0 \quad (7.11)$$

$$\rightarrow |\det DG(x)| m(Q) = \int_Q |\det DG(x)| dm \quad \text{as } \epsilon \rightarrow 0 \quad (7.12)$$

证明了这一结论, 我们就完成了这个 proof 的一大半.

Step 2: Prove

$$m(G(U)) \leq \int_U |\det DG(x)| dm$$

for open U 的 case.

Proof of Step 2: Directly follows from 上一 lecture 的这个 statement: 任意 open $E \subset \mathbb{R}^n$ 都是 countable disjoint interior cubes 的 union.

Step 3: Prove

$$m(G(E)) \leq \int_E |\det DG(x)| dm$$

for E Borel 的 case.

Proof of Step 3: Apply step 2 的结论, 使用 MCT for L^+ case, 使用 DCT for L^1 case. 至此, 我们完成了 (b) 的证明的一个方向, 由此可以完成 (a) 的不等式的一个方向:

Step 4: 证明

$$\int_{G(\Omega)} f dm \leq \int_{\Omega} f \circ G |\det DG(x)| dm$$

simple function 的 case reduces to measure, 而 L^+ 的 case follows from MCT.

Step 5: 不等式的另一方向: 其实很简单, 因为 diffeomorphism 的 inverse 仍然是 diffeomorphism, 所以 apply inverse 可得.

注意, 这只是 for Borel E 和 L^+ Borel measurable f , 不过我们容易接着推导出 Lebesgue measurable E 的情况和 $f \in L^+(m)$ 的情况; 从而再接着推导出 $f \in L^1(m)$ 的情况.

Remark 这个证明写得比较潦草. 详情见 Folland 2.47.

但是大概思路都比较简单. 其中比较困难的是 Step 1 中的各种 error bounds. 很麻烦.

7.2.2 application of COV: polar coordinate

Def 7.2 (mapping from Euclidean coord to polar coord)

我们定义:

$$\Phi : \mathbb{R}^n \setminus \{0\} \rightarrow (0, \infty) \times S^{n-1}$$

by:

$$x \mapsto (r \in \mathbb{R}, \theta \in S^{n-1})$$

其中,

$$r = |x|, \quad \theta = \frac{x}{|x|} \in S^{n-1}$$



这是一个很直观的坐标变换, 即一个 diffeomorphism.

Def 7.3 (a Borel measure on $(0, \infty) \times S^{n-1}$)

我们定义

$$m_*(E) := m(\Phi^{-1}(E))$$



这是一个通过坐标变换的 preimage 的 Borel measure 定义的新的 Borel measure.

Theorem 7.6

Define Borel measure ρ on $(0, \infty)$ by:

$$\rho(E) = \int_E r^{n-1} dr$$

存在 unique 的 Borel measure σ_{n-1} on S^{n-1} , 使得 for Borel measurable $f : \mathbb{R}^n \rightarrow \mathbb{C}$ 且 $f \geq 0$ or $f \in L^1(m)$, 有

$$\int_{\mathbb{R}^n} f(x) dm \stackrel{COV}{=} \int_{(0, \infty) \times S^{n-1}} f(r\theta) dm_* \quad (7.13)$$

$$\stackrel{Fubini}{=} \int_0^\infty \int_{S^{n-1}} f(r\theta) d\sigma d\rho \quad (7.14)$$

$$= \int_0^\infty r^{n-1} \int_{S^{n-1}} f(r\theta) d\sigma dr \quad (7.15)$$



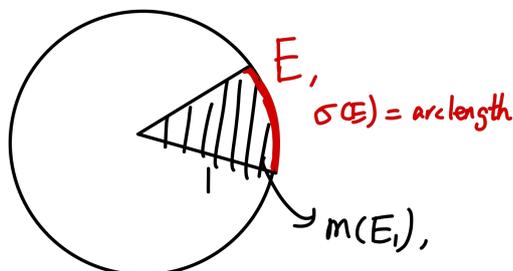
Proof 见 Folland 2.49.

Remark

这里 S^{n-1} 的 unique measure σ 的计算公式是:

$$\sigma(E) = n \cdot m\left(\Phi^{-1}((0, 1) \times E)\right) = n \cdot m\{r\theta \mid 0 < r \leq 1, \theta \in E\}$$

这很容易直观:



这里 $n = 2$, $m(E_1)$ 表示的单位圆下, E 的弧长下的扇形面积, 而 $\sigma(E)$ 表示 E 的 arc length.

(类比, 在 $n = 3$ 的情况下, $m(E_1)$ 表示单位球下, E 的球面下的锥形体积, $\sigma(E)$ 表示 E 在 S^2 中的球面面积.)

Remark 对于 $E = S^{n-1}$ 即全集的情况, 这个 measure 有固定的计算公式.

$$\sigma(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

Example 7.2 $\sigma(S^1) = 2\pi$, $\sigma(S^2) = 4\pi$.

Example 7.3 使用 polar coordinate 计算积分:

$$\int_{\mathbb{R}^n} e^{-a|x|^2} dx = \left(\frac{\pi}{a}\right)^{\frac{n}{2}}$$

这是因为:

$$I_2 = 2\pi \int_0^\infty r e^{-ar^2} dr = \frac{\pi}{a}$$

而由于

$$e^{-a|x|^2} = \prod_{j=1}^n e^{-ax_j^2}$$

我们得到

$$I_n = (I_1)^n$$

特别地,

$$I_2 = I_1^2, \quad \text{thus } I_1 = \left(\frac{\pi}{a}\right)^{\frac{1}{2}}$$